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A sufficient condition for planar graphs to be 3-colorable

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Abstract

Planar graphs without 3-cycles at distance less than 4 and without 5-cycles are proved to be 3-colorable. We conjecture that, moreover, each plane graph with neither 5-cycles nor intersecting 3-cycles is 3-colorable. In this conjecture, none of the two assumptions can be dropped because there exist planar 4-chromatic graphs without 5-cycles, as well as planar 4-chromatic graphs without intersecting triangles.

1. Introduction

As proved by Garey et al. [1], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, it seems wise to discuss only sufficient conditions for a planar graph to be 3-colorable. As early as 1969, Havel [3] asked if there existed a constant C such that each planar graph with the minimal distance between triangles at least C was 3-colorable. In 1976, Steinberg conjectured that each planar graph without 4- and 5-cycles was 3-colorable (see Problem 2.9 [4,5]). Both problems remain widely open. One of the purposes of this paper is to pose the following.

Conjecture 1.1. *Every planar graph without intersecting* 3-*cycles and without* 5-*cycles is* 3-*colorable*.

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This [Bordeaux 3-color] conjecture if true is best possible in the sense that there exist 4-chromatic planar graphs without intersecting triangles (as shown by Havel [3], see Fig. 1) and also those without 5-cycles (see Fig. 2).

We also pose the following [strong Bordeaux] conjecture, which implies both Conjecture 1.1 and Steinberg's one:

Conjecture 1.2. *Every planar graph without adjacent* 3*-cycles and without* 5*-cycles is* 3*-colorable.*

(By intersecting (adjacent) triangles we mean those with a vertex (an edge) in common.) Our second purpose is to make the first step in the direction of Conjectures 1.1 and 1.2.

Theorem 1.3. *Every planar graph with neither* 3*-cycles at distance less than* 4 *nor* 5*-cycles is* 3*-colorable.*

Informally speaking, this result shows that the "intersection" of Havel's and Steinberg's problems has a positive solution, i.e., each graph satisfying both Havel's and Steinberg's assumptions is 3-colorable. It also says that the main difficulty in Havel's problem is hidden in the 5-cycles. Finally, it is the first positive result on 3-coloring that involves the minimal distance between triangles.

It was easier for us to prove the following fact:

Theorem 1.4. *Every proper* 3*-coloring of a face of size* 3 *or* 7 *in a connected plane graph that has neither a pair of* 3*-cycles at distance less than* 4 *nor a* 5*-cycle can be extended to a proper* 3*-coloring of the whole graph.*

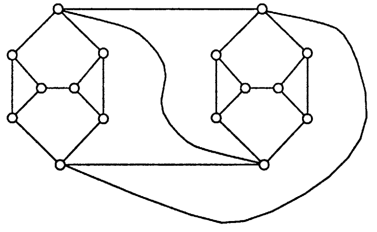


Fig. 1. A 4-chromatic graph without intersecting triangles.

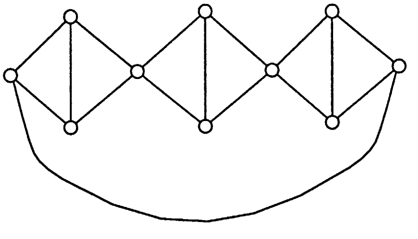


Fig. 2. A 4-chromatic graph without 5-cycles.

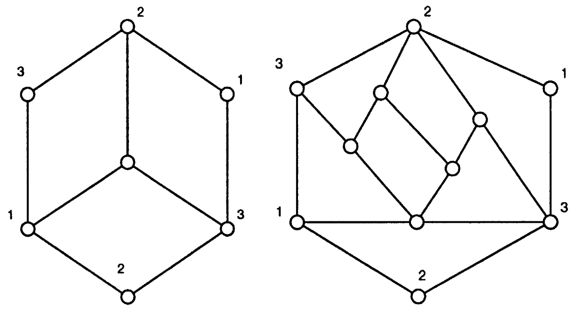


Fig. 3. Not extendable 3-coloring.

Fig. 3 represents two graphs such that the 3-coloring of the outside 6-face cannot be extended to the whole graph.

To deduce Theorem 1.3 from Theorem 1.4, suppose that G is a counterexample to Theorem 1.3 on the fewest vertices. Clearly, G is connected and by Grötzsch's theorem [2] has a 3-cycle, which is clearly a 3-face, f, by the minimality of G. By Theorem 1.4, a 3-coloring of f can be extended to a 3-coloring of G.

This formulation of Theorem 1.4 is not the strongest possible. As observed by a referee, (i) putting at least two vertices of degree 2 on an edge of the outside faces in Fig. 3 gives counterexamples with precolored faces of arbitrary size greater than 7 and, on the other hand, (ii) the problem of 3-color extension from a 4-face is reduced to that from a 7-face by putting three vertices of degree 2 on a boundary edge of the 4-face.

We denote the degree of a vertex v by $d(v)$ and the size of a face f by $r(f)$ (bridges are counted twice); a k-vertex is that of degree k, a $\geqslant k$-vertex has the degree at least k, etc.

2. Proof of Theorem 1.4

2.1. Properties of a minimal counterexample

Let T be the minimum number of 3-cycles in the counterexamples to Theorem 1.4, and let N be the minimum number of vertices in the counterexamples having T cycles of length 3. Finally, let G have the fewest edges among the counterexamples to Theorem 1.4 having T 3-cycles and N vertices.

Suppose that a proper 3-coloring φ of a face f_0 in G cannot be extended to a proper 3-coloring of G. W.l.o.g., suppose that f_0 is the outside face. Denote the boundary of f_0 by $D = d_1 d_2 \dots d_{|D|}$.

(1) f_0 is a 7-face. Otherwise, we put four vertices of degree 2 on a boundary edge of the 3-face f_0 and extend φ to the new vertices. The graph G' obtained has fewer

3-cycles, is connected, and has neither 3-cycles at distance less than 4 nor 5-cycles. It follows, φ can be extended to G', which gives also a proper 3-coloring of G.

(2) G is 2-connected; in particular, G has no 1-vertices. First suppose that the cycle D is not simple and d_1 is its cut-vertex. Then D consists of a 3-cycle $d_1d_6d_7$ and either (i) a 4-cycle $d_1d_2d_3d_4$, or (ii) two pendant edges (in two ways), or else (iii) a pendant 2-path. We put four 2-vertices on the edge d_1d_7 to obtain a 7-cycle C_3'. In case (i), we put three 2-vertices on the edge d_1d_2 to obtain a 7-cycle C_4'. Then we extend φ to C_3' and C_4'. Now we can extend φ to the interiors of C_3' and C_4' as each of them has fewer 3-cycles than G. This completes the proof in case (i); cases (ii) and (iii) are even easier.

Thus D is a simple cycle, and it is contained in a block of G. Hence, there is a pendant block B of G, with a cut-vertex v, such that $(B-v)\cap D=\emptyset$. The graph G' obtained from G by deleting $B-v$ has a 3-coloring that extends φ. In turn, B is 3-colorable, either by the Grötzsch theorem, or by the minimality of G, and we can choose a color for v in B the same as in G'. These two 3-colorings give a suitable 3-coloring of G.

(3) Each 2-vertex in G belongs to D.

(4) D is a simple 7-cycle without chords. Indeed, if D has a chord then, due to the absence of 5-cycles in G, this chord splits D into a 3-cycle and a 6-cycle. We delete the chord and extend φ to the remaining graph by the edge-minimality of G. The 3-coloring obtained is proper also for G.

(5) G has no separating 3-cycles and 7-cycles. If such a separating cycle S exists, we first extend φ from D to $G-Int(S)$, and then extend the obtained 3-coloring of the outside face S of $G-Ext(S)$ to the whole $G-Ext(S)$ (either by the minimality of G or by the Grötzsch theorem).

The following properties (5+) of G are easy consequences of (1–5) and will be used throughout the proof without reference.

(5+) (a) Every 3-cycle is actually a 3-face,
- (b) each simple 7-cycle is either a 7-face or consists of a 3-face adjacent to a 6-face,
- (c) no 3-face is adjacent to a 4-cycle, and
- (d) no 3-face can share two edges with another face.

(6) Each 4-face has a vertex in common with D. Suppose $f=wxyz$ is a face inside D. First observe that identifying x with z (or y with w) within f does not create a 3-cycle: this follows from the absence of 5-cycles in G.

Now suppose y is a vertex of f whose distance, $\rho(y)$, from the triangles of G is the least as compared to w, x and z, and let T be a triangle at distance $\rho(y)$ from y. Recall that f cannot share an edge with a 3-cycle. If $\rho(y)$ is at most 1, then the distance from x and z to T is $>\rho(y)$ due to the absence of 5-cycles in G (see Figs. 4 and 5). Hence

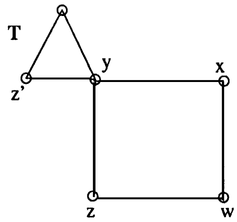


Fig. 4. $\rho(y) = 0 \Rightarrow z' \neq z$.

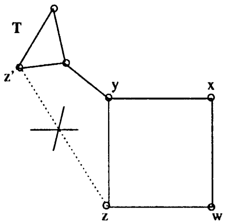


Fig. 5. $\rho(y) = 1 \Rightarrow zz' \notin E$.

$\rho(y) = 1$ implies $\rho(x) \geqslant 2$ and $\rho(z) \geqslant 2$, and if $\rho(y) = 0$ then each of x, z lies at distance 1 from T and at distance at least 3 from each other 3-face of G.

It follows that in both cases identifying x with z does not create a pair of triangles at distance less than 4.

We next show that identifying x with z does not create a 5-cycle $xv_1 \dots v_4$. Suppose the contrary, and by symmetry between y and w (the minimality of y is not used anymore) we can assume w.l.o.g. that y lies non-strictly inside the 7-cycle $zwxv_1 \dots v_4$. Due to (5), y must coincide with one of $v_1, \dots, v_4$, but this is impossible due to the absence of 5-cycles.

Hence, identifying x with z cannot create a 5-cycle or a pair of 3-cycles at distance less than 4. Also observe that since neither x nor z belongs to D by assumption, it follows that the coloring φ of D remains proper, because there are only two ways to spoil φ by contraction: (i) to identify two vertices of D colored differently, or (ii) to join by an edge two vertices of D colored the same.

So, identifying x with z yields a graph G^* which has a 3-coloring that extends φ, and this coloring trivially gives a 3-coloring of G that extends φ, a contradiction.

(7) G cannot have a 4-face sharing two consecutive edges with D. Suppose $f = wxyz$ is a 4-face such that $\{x, y, z\} \in D$. Then $w \notin D$ because of the absence of 5-cycles in G. We extend φ to w by putting $\varphi(w) = \varphi(y)$. Clearly, φ is still proper since w cannot be adjacent to a vertex from $D - \{x, z\}$ due to the absence of 5-cycles. So, φ can be extended from the outside face of $G - y$ to the whole $G - y$, which immediately gives a 3-coloring of G; a contradiction.

(8) G cannot have a 4-face $f = wxyz$ sharing precisely one edge, xy, with D. Suppose that $x = d_3$ and $y = d_2$. First observe that $z \notin D$: Due to (7), $z \neq d_1$, and if $z = d_i$ then we have a 5-cycle whenever $5 \leqslant i \leqslant 7$ and a separating 3-cycle for $i = 4$. By symmetry, $w \notin D$.

Due to the same argument as in proving (6), we can assume that identifying x with z does not create a 5-cycle nor a pair of 3-cycles at distance less than 4. To obtain a contradiction, it remains to show that this operation leaves the coloring φ on D proper.

Since z is inside D, we should only exclude the possibility to make φ improper because of the edge zd_t. By (5+), $1 \neq t \neq 3$. Suppose $t = 7$. Then we have a 7-cycle $d_4 \dots d_7 zyx$, which cannot be separating due to (5). It follows that $w \in D$, a contradiction. Clearly, $6 \neq t \neq 5$ because of the absence of 5-cycles. Finally, if $t = 4$, then identifying z with x does not actually create new adjacencies due to the presence of the edge xd_4 (we simply delete the edge zd_4).

Thus, we can extend φ to the graph with x and z identified, and hence to G itself; a contradiction.

(9) If a 3-vertex y is incident with a 6-face $f = xyzw_1w_2w_3$ and a 3-face $f' = xyt$, where $\{x, y, z\} \cap D = \emptyset$, then:

(a) $d(t) > 3$, and
(b) either $d(x) > 3$ or $t \in D$.
We first prove that
(c) the third face $f'' = tyz\dots$ incident with y has size 6.

To this end, let us observe that deleting y followed by identifying x with z does not create a 3-cycle due to the absence of 5-cycles in G. Next, observe that due to the presence of the 3-face f', this operation does not create two 3-cycles at distance less than 4. Since both x and z are inside D, their identification does not affect the coloring φ of D. Thus, the only obstacle for identifying x with z is creating a 5-cycle $xv_4 \dots v_1$. Suppose this has happened and consider the two cases.

Case 1: The path $w_1w_2w_3$ lies non-strictly inside the 7-cycle $xyzv_1 \dots v_4$. Due to (5), all w_i's, where $1 \leqslant i \leqslant 3$, must coincide with some of v_j's, where $1 \leqslant j \leqslant 4$. Clearly, $w_1 \neq v_2$ because of the absence of two 3-faces at distance at most 1. If w_1 coincides with v_3 or v_4, we have a 5-cycle. Hence, $w_1 = v_1$, and, by symmetry, $w_3 = v_4$. Then w_2 coincides with v_2 or v_3, and we have a 3-cycle too close to f'; a contradiction.

Case 2: t lies non-strictly inside the 7-cycle $xyzv_1 \dots v_4$.

Similarly, t must coincide with one of $v_1, \dots, v_4$. However, $t \neq v_4$ readily implies the presence of two adjacent 3-faces or a 5-cycle in G.

This completes the proof of (c).

Now we have a symmetric configuration shown in Fig. 6. To prove (a), suppose $d(t) = 3$ (by (2–4), $d(t) \geqslant 3$). Then t cannot lie in D, for otherwise x would be in D.

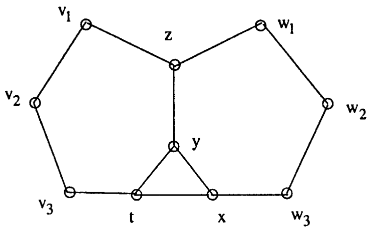


Fig. 6. Equation (9).

As follows from proving (9c), the only non-separating 7-cycle going through x and z goes also through t. We thus delete t along with y and destroy this last obstacle for identifying x with z.

So, identifying x with z followed by deleting y and possibly t, yields a graph which has a 3-coloring that extends φ, and this coloring trivially gives a 3-coloring of G that extends φ: if t was deleted, we first color t and then y; otherwise we just color y.

This completes the proof of (a).

To prove (b), suppose that $t \notin D$. Then we apply (9) symmetrically, i.e., to the face f'', which is a 6-face as proved in (c), rather than to f. (Now the roles of t and x are interchanged.) It follows from the symmetric version of (a) that $d(x) > 3$.

2.2. No minimal counterexample can exist

The rest of our proof consists in proving that the structural properties (1–9) of G are mutually incompatible. Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ for G may be rewritten as

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (r(f) - 4) = -8.$$

We set the *initial charge* of every vertex v of G to be $ch(v) = d(v) - 4$, that of every face $f \neq f_0$ to be $ch(f) = r(f) - 4$, and also set $ch(f_0) = r(f_0) + 4$. Clearly,

$$\sum_{x \in V(G) \cup F(G)} ch(x) = 0.$$

Now we use the discharging procedure, leading to the *final charge* ch^*, defined by applying the rules R0–R5 below. Since this procedure preserves the total charge, we have

$$\sum_{x \in V(G) \cup F(G)} ch^*(x) = 0.$$

The rest of our proof will consist in showing that $ch^*(x) \geqslant 0$ whenever $x \in V(G) \cup F(G)$, and $ch^*(f_0) > 0$, with the obvious final contradiction $(0 > 0)$.

R0. Each 3-face $f = xyz$ receives $\frac{1}{3}$ from each adjacent face, unless $d(x) = 3$, $d(y) \geqslant 4$, and $d(z) \geqslant 4$, in which case f receives from adjacent faces $\frac{1}{6}$ across each of the edges xy, xz and $\frac{2}{3}$ across yz.

R1. Every 3-vertex $v \notin D$ receives $\frac{1}{3}$ from each incident face, unless v is incident with one 3-face, in which case v receives $\frac{1}{2}$ from each of the two >3-faces.

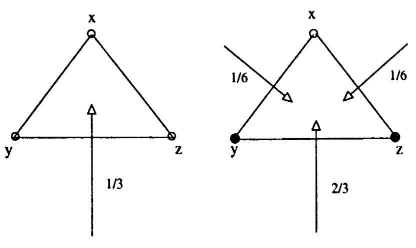
R2. Every 2-vertex receives $\frac{5}{3}$ from f_0 and $\frac{1}{3}$ from the other (internal) incident face.

R3. f_0 gives 1 to every incident vertex of degree at least 3.

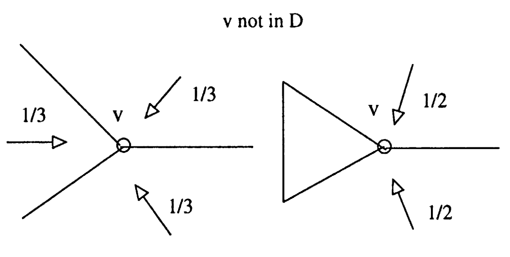
R4. Every vertex $d_2 \in D$ with $d(d_2) \geqslant 4$ gives 1 to each incident face not incident with the edges $d_1 d_2$ and $d_2 d_3$. Furthermore, if 1 is given to a 3-face $f = d_2 xy$ such that $x \notin D$, $y \notin D$, then this 1 is transferred by f across the edge xy to the neighbor face.

R5. Each $\geqslant$8-face $f \neq f_0$ gives $(r(f) - 8)/2$ to f_0.

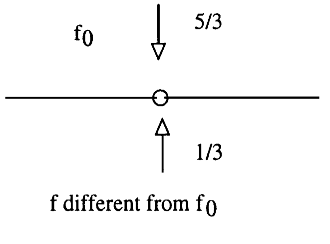
R0.



R1.



R2.



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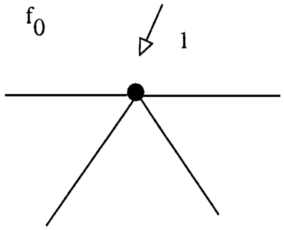
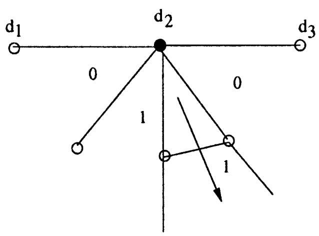
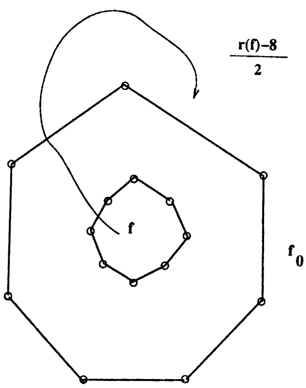


R4.



R5.



(10) If $v \in V(G)$ then $ch^*(v) \geqslant 0$.

First suppose $v \notin D$. If $d(v) = 3$, then by R1 $ch^*(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$ when v is not incident with a 3-face and $ch^*(v) = 3 - 4 + 2 \times \frac{1}{2} = 0$ otherwise. If $d(v) \geqslant 4$, then $ch^*(v) = ch(v) = d(v) - 4 \geqslant 0$.

Now suppose $v \in D$. If $d(v) = 2$ then $ch^*(v) = 2 - 4 + \frac{1}{3} + \frac{5}{3} = 0$ by R2. If $d(v) \geqslant 3$, then v receives 1 from f_0 by R3 and sends away $d(v) - 3$ units of charge by R4, so that $ch^*(v) = d(v) - 4 + 1 - 1 \times (d(v) - 3) = 0$.

(11) If $f \in F(G)$ and $f \neq f_0$, then $ch^*(f) \geqslant 0$.

If $r(f) = 3$, then its initial charge is -1 and f receives either $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ or $\frac{2}{3} + \frac{1}{6} + \frac{1}{6}$ by R0, so that $ch^*(f) \geqslant 0$. (If f appears in R4, then f can only win from that in total.)

If $f = wxyz$, i.e., $r(f) = 4$, then its initial charge is 0. By (6), f has a vertex, say y, in common with D. Then neither x nor z belongs to D due to (8) and (4). It follows that $d(y) \geqslant 4$ and f receives 1 from y by R4. After giving $\frac{1}{3}$ by R1 to at most three incident vertices, f has $ch^*(f) \geqslant 0$. (By the absence of 5-cycles, f cannot give $\frac{1}{2}$ according to R1 or give anything to a neighbor 3-face by R0.)

We now consider the most difficult case $r(f) = 6$. Suppose $f = v_1 \dots v_6$ and recall that the initial charge of f is 2. Also recall that $f \neq f_0$ does not give anything to vertices of degree greater than 2 from D. If our f is not adjacent to a 3-face, then f gives at most $\frac{1}{3}$ to at most six vertices by R1, R2 so that $ch^*(f) \geqslant 2 - \frac{1}{3} \times 6 = 0$.

So suppose there is a 3-face $f' = tv_1v_2$. Such a face is unique, and it has only one edge v_1v_2 in common with f. If each of v_1 and v_2 has degree greater than 3 or belongs to D (and thus has degree at least 3, being incident with a 3-face), then f only gives $\frac{2}{3}$ across v_1v_2 by R0 and at most $\frac{1}{3}$ to at most four vertices by R1, and we are done.

We can thus assume that $d(v_2) = 3$ and $v_2 \notin D$. Now f sends at most $\frac{1}{3}$ across v_1v_2 and gives $\frac{1}{2}$ to v_2. The total donation of f in the worst case cannot exceed $\frac{8}{3}$: $\frac{1}{2}$ to v_1 and $\frac{1}{3}$ to each of the four other incident vertices.

Let us first prove $ch^*(f) \geqslant 0$ assuming that at least one of the vertices v_1, v_3 belongs to D. Suppose $v_1 \in D$ (our argument makes no difference between v_1 and v_3). It suffices to show that either there is a $v_i \in D$ giving 1 to f or there are two $v_i \in D$'s taking 0 each from f: then the total donation of f is in fact at most $\frac{8}{3} - \frac{2}{3} = 2$. If $v_6 \notin D$, then f receives 1 from v_1 by R4, and we are done. Suppose $v_6 \in D$. Then v_1, being adjacent to $v_2 \notin D$ and thus having degree at least three, gets nothing from f. Another vertex getting nothing from f is the first among $v_6, v_5, \dots$ that has a neighbor inside D, and we are done again.

Now suppose that neither v_1 nor v_3 belongs to D. Then $d(t) \geqslant 4$ by (9a). By (9b), either $d(v_i) \geqslant 4$ or $t \in D$. In the first case, f gives $\frac{1}{6}$ to tv_1v_2 by R0 and nothing to v_1. It follows that $ch^*(f) \geqslant 2 - \frac{1}{6} - \frac{1}{2} - \frac{1}{3} \times 4 = 0$. In the second case, f receives 1 across the edge v_1v_2 due to the second part of R4.

This completes the proof for $r(f) = 6$.

The next case is $r(f) = 7$. Now f still is adjacent to at most one 3-face, and so can give at most $\frac{2}{3}$ to a 3-face and at most $\frac{1}{2} \times 2 + \frac{1}{3} \times 5$ to the vertices, i.e., $\frac{10}{3}$ in total. However, giving $\frac{2}{3}$ across an edge implies giving nothing to the two incident vertices of degree at least 4, so in fact $ch^*(f) \geqslant 0$.

Now suppose $f = v_1 \dots v_r$, where $r \geqslant 8$. We want to partition all the donation of f to the vertices by R1, R2 and across the edges by R0 into r groups, associated with the incident vertices of f, so that the total donation per group were at most $\frac{1}{2}$. Along with the donation of $(r-8)/2$ to f_0 by R5, this will yield $ch^*(f) \geqslant r - 4 - r/2 - (r - 8)/2 = 0$.

Formally, we define these groups as follows. If f gives $\frac{2}{3}$ across the edge v_iv_{i+1} (addition is mod r), we split this donation of $\frac{2}{3}$ evenly between the vertices v_i and v_{i+1}, which in this case do not receive anything from f. If f gives $\frac{1}{3}$ (or $\frac{1}{6}$) across the edge v_iv_{i+1} by R0, we split this $\frac{1}{3}$ evenly between v_{i-1} and v_{i+2}, which receive at most $\frac{1}{3}$ each from f by R1, R2 (or, respectively, send $\frac{1}{6}$ to v_{i-1}). It remains to observe that every two crossed edges lie at distance at least 4 along the boundary of f.

(12) $ch^*(f_0) > 0$.

Recall that $D = d_1 \dots d_7$ and $ch(f_0) = r(f_0) + 4 = 11$. We show that f_0 either sends away less than 11 in total or loses precisely 11 but acquires at least $\frac{3}{2}$ by R5.

By R2, f_0 gives $\frac{5}{3}$ to each incident 2-vertex. By R3, f_0 gives 1 to each incident $\geqslant$3-vertex. Since $G \neq C_7$ and G is 2-connected, it follows that D has at least two $\geqslant$3-vertices. The total donation of f_0 is thus at most $\frac{2}{3} + 1 \times 2 + \frac{5}{3} \times 5 = 11$, i.e., $ch^*(f_0) \geqslant 0$. Moreover, the equality can hold only if D is adjacent to a 3-face and incident with five 2-vertices. But then the only non-triangular face f adjacent to f_0 has at least four internal vertices due to the absence of 5-cycles and intersecting 3-faces in G. Thus $r(f) \geqslant 7 + 4 = 11$, so that f sends to f_0 at least $(11 - 8)/2 > 0$ by R5.

Due to the remark before formulating rules R0–R5, this completes the proof of Theorem 1.4.

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References

[1] M.R. Garey, D.S. Johnson, L.J. Stockmeyer, Some simplified NP-complete graph problems, Theoret. Comput. Sci. 1 (1976) 237–267.
[2] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin Luther Univ. Halle Wittenberg, Math.-Nat. Reihe 8 (1959) 109–120.
[3] I. Havel, On a conjecture of B. Grünbaum, J. Combin. Theory 7 (1969) 184–186.
[4] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley-Interscience, New York, 1995.
[5] R. Steinberg, The state of the three color problem, in: J. Gimbel, J.W. Kennedy, L.V. Quintas (Eds.), Quo Vadis, Graph Theory? Ann. Discrete Math. 55 (1993) 211–248.